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On output feedback control of singularly perturbed systems

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ABSTRACT

We treat the problem of robustness of output feedback controllers with respect to singular perturbations. Given a singularly perturbed control system whose boundary layer system is exponentially stable and whose reduced order system is exponentially stabilizable via a (possibly dynamical) output feedback controller, we present a sufficient condition which ensures that the system obtained by applying the same controller to the original full order singularly perturbed control system is exponentially stable for sufficiently small values of the perturbation parameter. This condition, which is less restrictive than those previously given in the literature, is shown to be always satisfied when the singular perturbation is due to the presence of fast actuators and/or sensors. Furthermore, we show explicitly that, in the linear time-invariant case, if this condition is not satisfied then there exists an output feedback controller which stabilizes the reduced order system but destabilizes the full order system.

1. Introduction

The problem of control design for singularly perturbed systems has attracted the attention of many researchers for many years. This is due not only to theoretical interest but also to the relevance of this topic in control engineering applications. Indeed multiple time scale phenomena are almost unavoidable in "real life" systems and the singular perturbation approach has proven to be a powerful tool for system analysis and control design.

In this paper, we use singular perturbation theory to study the following problem. Suppose one has a system with two time scale dynamics: slow dynamics, those of interest for control engineering, and fast "parasitic" dynamics, and suppose that it is required to stabilize the system via (possibly dynamical) output feedback. If the fast dynamics are asymptotically stable, a "rule of thumb" for approaching this kind of design problem is to consider the fast dynamics as instantaneous, that is to neglect them, and design the controller for the *reduced order* system thus obtained. This relies on the fact that, under mild technical conditions, asymptotic stability of both the reduced order system and the boundary layer system implies asymptotic stability of the full order system for sufficiently small values of the perturbation parameter (see, for example, [10,17]). The order reduction implied by this procedure can greatly simplify the design task, but questions arise on the "robustness" of the procedure: Will the controller designed for the simplified model stabilize the system in presence of fast dynamics which are not instantaneous? Since the controller may destabilize the full order system, in general output feedback controllers lack the robustness property which is instead inherent to "slow"-state feedback controllers. Robustness, in the singular perturbation sense, can be achieved only by placing some constraints on the system and/or the controller.

The problem has been extensively studied for linear time-invariant (LTI) systems in [14,15]. (The results in the first paper can be found also in [17].) There it is shown that sufficient conditions for robustness of output feedback are that either (i) the

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input does not act on the singularly perturbed equation; or (ii) the output does not depend on the "fast"-state; or (iii) there is no static feedback from the output to the input. In [22] a condition is given which ensures the convergence of the full order system to the reduced order system in the so-called *graph topology* and this in turn guarantees the robustness of any controller whose design is based on the reduced order system. In [23] the problem of robustness is analysed from a slightly different point of view. Refs. [3,9,16,18,19] are related to the issue of output feedback for singularly perturbed LTI systems, but not directly to that of robustness. In the framework of singularly perturbed nonlinear systems, sufficient conditions for robustness of output feedback control have been given in [8,2]. More recently, Christofides [20] and Khalil [21] use high gain observer based controllers to achieve robust output feedback controllers for specific classes of nonlinear systems. In [20], the output does not depend on the system fast state and in [21] the singular perturbation is due to linear actuator and sensor dynamics.

Here, dealing with a general class of singularly perturbed nonlinear systems, we give a sufficient condition which guarantees robustness of output feedback controllers. In the LTI case this condition is less restrictive than those previously stated in the literature and it is equivalent to that one given in [22]. Moreover it is shown that, in the nonlinear case, this condition is always satisfied when the singular perturbation of the system is due to the presence of fast actuators and/or sensors: this is an important result for applications. Finally, we show that, in the LTI case, if this condition is not satisfied then there exists an output feedback controller which stabilizes the reduced order system but destabilizes the full order system.

1.1. Notation

The following notation will be employed in this paper.

$\mathbb{R}(\mathbb{R}_+)$	The set of (nonnegative) real numbers
$\mathbb{C}$	The set of complex numbers
I_n	The $n \times n$ identity matrix (the subscript will be dropped when n is clear from the context)
u_i	The ith component of the vector u
$\mathscr{R}(A)(\mathscr{N}(A))$	The range (null space) of the matrix A
$D_i f$	The "block" partial derivative of the function f wrt its ith argument [1, p. 360]
$L_g h$	The directional derivative of the scalar function h along the direction of the vector field g, that is, $L_g h \triangleq Dhg$
$O(\mu)$	Given a function $f : \mathbb{R}_+ \to X$, with X a normed linear space, we write $f(\mu) \sim O(\mu)$ if $\|f(\mu)\|/\mu$ is bounded for $\mu > 0$ sufficiently small

We will frequently use the following fact. If $f : \mathbb{R}^p \to \mathbb{R}^q$ is continuously differentiable and $\|Df(x)\| \leqslant M$ for all $x \in \mathbb{R}^p$ then,

$$\|f(y) - f(x)\| \leqslant M\|y - x\|$$

for all $x, y \in \mathbb{R}^p$ [1, Corollary 40.6].

Arguments of functions will sometimes be omitted when this is not likely to cause confusion.

2. Robustness of output feedback wrt singular perturbations

Consider a singularly perturbed input–output system described by

$$\dot{x} = F(t, x, z, u, \mu), \tag{1a}$$

$$\mu\dot{z} = G(t, x, z, u, \mu), \tag{1b}$$

$$y = V(t, x, z, \mu), \tag{1c}$$

where $t \in \mathbb{R}$ is called the time variable, $x(t) \in \mathbb{R}^n$ is called the system slow state, $z(t) \in \mathbb{R}^m$ is called the system fast state, $u(t) \in \mathbb{R}^p$ is the control input, $y(t) \in \mathbb{R}^l$ is the measured output and $\mu > 0$ is the singular perturbation parameter. We suppose that for some $\bar{\mu} > 0$, the functions $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times [0, \bar{\mu}) \to \mathbb{R}^n$, $G : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times [0, \bar{\mu}) \to \mathbb{R}^m$ and $V : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times [0, \bar{\mu}) \to \mathbb{R}^l$ are continuous.

The reduced order system. First, we make the following assumption.

Assumption 1. For each $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $u \in R^p$, the equation

$$G(t, x, \bar{z}, u, 0) = 0$$

has a unique solution $\bar{z} = H(t, x, u)$.

The above assumption guarantees that there exists a unique reduced order system associated with the original full order system (1); this system, obtained by letting $\mu = 0$, is given by

$$\dot{x} = \overline{F}(t, x, u), \tag{2a}$$

$$y = \overline{V}(t, x, u), \tag{2b}$$

where

$$\overline{F}(t,x,u) \triangleq F(t,x,H(t,x,u),u,0), \tag{3}$$

$$\overline{V}(t,x,u) \triangleq V(t,x,H(t,x,u),0). \tag{4}$$

The open loop boundary layer system. Consider any time t_0 and define a new time variable (sometimes called "fast time") by $\tau \triangleq (t - t_0)/\mu$. Substituting in (1) and letting $\mu = 0$, one obtains the open loop boundary layer system:

$$\frac{dz_f}{d\tau} = G(t_0, x_0, z_f, u_0, 0), \tag{5}$$

where $z_f(\tau) \in \mathbb{R}^m$ is the boundary layer state, $x_0 \triangleq x(t_0)$ and $u_0 \triangleq u(t_0)$. This is a parameterized dynamical system with (t_0,x_0,u_0) as parameter. Also, as a consequence of the previous assumption, it has a unique equilibrium state $z_{fe} = H(t_0, x_0, u_0)$.

Before stating the next assumption, we need a definition. Consider a parameterized system described by

$$\dot{\xi} = S(t, \xi, \theta), \tag{6}$$

where $\xi(t) \in \mathbb{R}^n$ with $t \in \mathbb{R}$, and θ is a parameter belonging to some set Θ. Suppose that for each θ this system has an equilibrium state $\xi_e(\theta)$ and let $\phi(\cdot; t_0, \xi_0, \theta)$ denote any solution of this system corresponding to the initial condition $\xi(t_0) = \xi_0$.

Definition 1. The parameterized system (6) is globally uniformly exponentially stable (GUES) about $\xi_e(\theta)$ with rate of convergence α and *gain* β if for all $\theta \in \Theta$, $t_0 \in \mathbb{R}$ and $\xi_0 \in \mathbb{R}^n$,

$$\|\phi(t; t_0, \xi_0, \theta) - \xi_e(\theta)\| \leqslant \beta \|\xi_0 - \xi_e(\theta)\| e^{-\alpha(t-t_0)} \tag{7}$$

for all $t \geqslant t_0$. The supremum of all the rates of convergence is called the supremal rate of convergence.

Remark 1. It is readily seen that if system (6) is GUES about $\xi_e(\theta)$ then, for each θ, $\xi_e(\theta)$ is an unique equilibrium state.

Assumption 2. The open loop boundary layer system (5) (considered as a parameterized system with (t_0,x_0,u_0) as parameter) is GUES about $H(t_0,x_0,u_0)$.

The question considered here is that of *robustness of output feedback stabilization wrt singular perturbations*. Precisely, we ask under what conditions every (possibly dynamical) output feedback controller, which stabilizes the reduced order system, also stabilizes the original full order system, provided the perturbation parameter μ is sufficiently small.

It is not difficult to see that problems can arise whenever the fast state z is statically fed-back to Eq. (1b) via the output equation, since this feedback may alter the boundary layer system associated with the closed loop system and make it unstable; see next example. Thus, a general guideline for robust output feedback stabilization of a singularly perturbed system has been to avoid static feedback; see [17, p. 146]. In the framework of LTI systems, a detailed analysis of this has been carried out in [14,15,17].

Example 1. As an illustrative example, consider the following simple singularly perturbed system

$$\begin{aligned} \dot{x}_1 &= z_1, \\ \mu\dot{z}_1 &= z_2, \\ \mu\dot{z}_2 &= x_1 - z_1 - z_2 + u, \\ y &= x_1 - z_2, \end{aligned} \tag{8}$$

where all variables are scalars. The reduced order system is given by

$$\begin{aligned} \dot{x}_1 &= x_1 + u, \\ y &= x_1, \end{aligned} \tag{9}$$

and the open loop boundary layer system is given by

$$\begin{aligned} \frac{dz_{f1}}{d\tau} &= z_{f2}, \\ \frac{dz_{f2}}{d\tau} &= -z_{f1} - z_{f2} + x_{10} + u_0. \end{aligned} \tag{10}$$

It is readily seen that the reduced order system is unstable for $u = 0$ and the open loop boundary layer system is exponentially stable; thus it is quite "natural" to design a stabilizing controller for the reduced order system and apply it to the original full order system (8).

All linear, static, output feedback, stabilizing controllers for the reduced order system are given by

$$u = -ky, \quad k > 1.$$

Applying this output feedback control law to the full order system results in the closed loop full order system,

$$\begin{aligned}\dot{x}_1 &= z_1,\\ \mu\dot{z}_1 &= z_2,\\ \mu\dot{z}_2 &= -(k-1)x_1 - z_1 - (1-k)z_2.\end{aligned} \tag{11}$$

Since the characteristic polynomial associated with this system is given by

$$p(s) = s^3 + \mu^{-1}(1-k)s^2 + \mu^{-2}s + \mu^{-2}(k-1),$$

we see that the closed loop full order system is unstable for any k which results in exponential stability of the closed loop reduced order system. Note that the boundary layer system associated with the closed loop full order system is unstable for $k > 1$.

Linear time-invariant (LTI) systems. A linear time-invariant singularly perturbed system is described by

$$\dot{x} = A_{11}x + A_{12}z + B_1u, \tag{12a}$$

$$\mu\dot{z} = A_{21}x + A_{22}z + B_2u, \tag{12b}$$

$$y = C_1x + C_2z, \tag{12c}$$

where $A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, C_1, C_2$ are matrices of appropriate dimensions. Assumption 1 is equivalent to the requirement that *the matrix A_{22} is nonsingular*. In this case, the reduced order system is described by

$$\dot{x} = \overline{A}x + \overline{B}u, \tag{13a}$$

$$y = \overline{C}x + \overline{D}u, \tag{13b}$$

where

$$\begin{aligned}\overline{A} &= A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad \overline{B} = B_1 - A_{12}A_{22}^{-1}B_2,\\ \overline{C} &= C_1 - C_2A_{22}^{-1}A_{21}, \quad \overline{D} = -C_2A_{22}^{-1}B_2.\end{aligned}$$

Assumption 2 is equivalent to the requirement that *the matrix A_{22} is Hurwitz*, that is, all the eigenvalues of A_{22} have negative real parts.

Static output feedback is robust if a condition is placed on the structure of the fast dynamics equation of the singularly perturbed system. For example, one can see that, if $B_2 = 0$ or $C_2 = 0$, then stability of the boundary layer system associated with the closed loop system is maintained under any output feedback. This guarantees that any linear controller which stabilizes the reduced order system (13b) also stabilizes the full order system for small μ. We will show in the following sections that a much stronger result holds.

3. The boundary layer input–output system

In order to state the next important assumption, we introduce a further definition. Consider an input–output system described by

$$\dot{\xi} = S(t, \xi, u), \tag{14a}$$

$$y = R(t, \xi), \tag{14b}$$

where $\xi(t) \in \mathbb{R}^n$ is the state at time $t \in \mathbb{R}$, $u(t) \in \mathbb{R}^p$ is the input, and $y(t) \in \mathbb{R}^l$ is the output. Suppose that corresponding to every initial condition $\xi(t_0) = \xi_0$ and every continuous input $u(\cdot)$, this system admits a unique solution, and let $\psi(t, t_0, \xi_0, u(\cdot))$ be the corresponding output at time t.

Definition 2. The input–output system (14b) is output invariant under the input [11] or input–output decoupled [25] if, for all $t,\ t_0 \in \mathbb{R}$ and $\xi_0 \in \mathbb{R}^n$,

$$\psi(t, t_0, \xi_0, u(\cdot)) = \psi(t, t_0, \xi_0, 0) \tag{15}$$

for all continuous input functions $u(\cdot)$.

The boundary layer input–output system. The boundary layer input–output system associated with the full order system (1) is the following parameterized input–output system:

$$\frac{dz_f}{d\tau} = G(t_0, x_0, z_f, u_f, 0), \tag{16a}$$

$$y_f = V(t_0, x_0, z_f, 0), \tag{16b}$$

where $z_f(\tau) \in \mathbb{R}^n$ is the state at (fast) time $\tau \in \mathbb{R}$, $u_f(\tau) \in \mathbb{R}^p$ is the input, $y_f(\tau) \in \mathbb{R}^l$ is the output, and $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ is the parameter vector.

Assumption 3. For each $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, the boundary layer input–output system (16b) is input–output decoupled.

In Example 1, the boundary layer input–output system is given by

$$\frac{dz_{f1}}{d\tau} = z_{f2}, \tag{17a}$$

$$\frac{dz_{f2}}{d\tau} = -z_{f1} - z_{f2} + u_f + x_{10}, \tag{17b}$$

$$y_f = -z_{f2} + x_{10}. \tag{17c}$$

Clearly, this does not satisfy the above assumption.

At first glance, the above assumption appears quite strong; however, it holds for a large and interesting class of singularly perturbed systems, namely those whose perturbation is due to the presence of fast actuators and/or sensors. We discuss this at length in Section 7. Here we note that Assumption 3 is trivially satisfied if the output map V does not depend on the fast state z, or if the function G does not depend on the input u.

3.1. Strict properness of the reduced order system

Note that the original full order system (1) is strictly proper in the sense that the output map V does not depend on the input u. Here we will show that, under Assumptions 2 and 3, the reduced order system is also strictly proper, that is, the output map $\overline{V}$ for the reduced order system is constant wrt u. To this end, consider any $t_0 \in \mathbb{R},\ x_0 \in \mathbb{R}^n$ and $u_0 \in \mathbb{R}^p$. Applying the constant input $u_f(\tau) \equiv u_0$ to the boundary layer input–output system, we obtain

$$\frac{dz_f}{d\tau} = G(t_0, x_0, z_f, u_0, 0),$$

$$y_f = V(t_0, x_0, z_f, 0).$$

Utilizing Assumption 2, every solution of this system satisfies

$$\lim_{\tau\to\infty} z_f(\tau) = H(t_0, x_0, u_0);$$

hence, recalling definition (4) of $\overline{V}$, every corresponding output y_f satisfies

$$\lim_{\tau\to\infty} y_f(\tau) = V(t_0, x_0, H(t_0, x_0, u_0), 0) = \overline{V}(t_0, x_0, u_0).$$

If the boundary layer input–output system is input–output decoupled then the output response to initial condition $z_f(0) = 0$ is the same for every constant input; hence $\overline{V}(t_0, x_0, u_0)$ is the same for every u_0. that is, the function $\overline{V}$ is constant wrt u. So, we can define

$$\bar{v}(t, x) \triangleq \overline{V}(t, x, 0) = V(t, x, H(t, x, 0), 0), \tag{18}$$

and the reduced order system can be rewritten as follows

$$\dot{x} = \overline{F}(t, x, u), \tag{19a}$$

$$y = \bar{v}(t, x). \tag{19b}$$

We have just proven the following lemma.

Lemma 1. *If Assumptions 2 and 3 hold then the reduced order system (2) is strictly proper.*

Remark 2. The hypotheses of Lemma 1 can be relaxed to Assumption 3 only. However, the proof of this more general result is less straightforward.

4. Main result

We will demonstrate robustness for stabilizing output feedback controllers of any order. First we consider static output controllers of the form:

$$u = k(t, y). \tag{20}$$

Later we consider dynamic controllers. This results in the closed loop reduced order system described by

$$\dot{x} = \bar{f}(t, x) \tag{21}$$

with

$$\bar{f}(t, x) \triangleq \overline{F}(t, x, k(t, \bar{v}(t, x))). \tag{22}$$

Assumption 4. There exists an output feedback controller $k : \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^p$ such that, the resulting closed loop reduced order system (21) is GUES about the origin.

The next two assumptions just place some technical conditions on the functions F, G, V, H and k. The following definition is useful in presenting these assumptions.

Definition 3. A function $f : \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_k} \to \mathbb{R}^q$ is nice if it is continuously differentiable,

$$f(t,0,\ldots,0) \equiv 0$$

and there exists a nonnegative constant M such that

$$\|D_1 f(t,\xi_1,\xi_2,\ldots,\xi_k)\| \leqslant M(\|\xi_1\| + \|\xi_2\| + \cdots + \|\xi_k\|),$$
$$\|D_{i+1} f(t,\xi_1,\xi_2,\ldots,\xi_k)\| \leqslant M \quad \text{for } i = 1,2,\ldots,k$$

for all $(t,\xi_1,\xi_2,\ldots,\xi_k) \in \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_k}$.

Remark 3. Consider any function f given by

$$f(t,\xi_1,\ldots,\xi_k) = A_1(t)\xi_1 + \cdots + A_k(t)\xi_k,$$

where each $A_i(\cdot),\ i = 1,\ldots,k$ is bounded and is continuously differentiable with bounded derivative. Then f is nice.

Remark 4. It is not difficult to prove that, if $f_1 : \mathbb{R} \times \mathbb{R}^{n_1} \to \mathbb{R}^q$ and $f_2 : \mathbb{R} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ are nice functions then, the composite function $f_1(\cdot,f_2)$ is also nice. Also, if a function $f : \mathbb{R} \times \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \to \mathbb{R}^q$ is nice, then the following inequality holds globally:

$$\|f(t,\xi_1,\ldots,\xi_k)\| \leqslant M(\|\xi_1\| + \cdots + \|\xi_k\|).$$

Assumption 5. The functions $F(\cdot,\cdot,\cdot,\cdot,0),\ G(\cdot,\cdot,\cdot,\cdot,0),\ V(\cdot,\cdot,\cdot,0),\ H$ and k are nice.

We point out that, in view of Assumption 4, assuming $F(t,0,0,0,0) \equiv 0,\ G(t,0,0,0,0) \equiv 0,\ V(t,0,0,0) \equiv 0$ and $k(t,0) \equiv 0$, does not constitute a loss of generality. Furthermore, $H(t,0,0) \equiv 0$ follows from $G(t,0,0,0,0) \equiv 0$ and Assumption 1.

Assumption 6. There exist a nonnegative constant d such that

$$\|F(t,x,z,u,\mu) - F(t,x,z,u,0)\| \leqslant \mu d(\|x\| + \|z\| + \|u\|),$$

$$\|G(t,x,z,u,\mu) - G(t,x,z,u,0)\| \leqslant \mu d(\|x\| + \|z\| + \|u\|),$$

$$\|V(t,x,z,\mu) - V(t,x,z,0)\| \leqslant \mu d(\|x\| + \|z\|)$$

for all $t \in \mathbb{R},\ x \in \mathbb{R}^n,\ z \in \mathbb{R}^m,\ u \in \mathbb{R}^p$ and $\mu \in (0,\bar{\mu})$.

Consider now the closed loop full order system obtained by applying the output feedback controller (20) to the full order system (1):

$$\dot{x} = f(t,x,z,\mu), \tag{23a}$$
$$\mu\dot{z} = g(t,x,z,\mu) \tag{23b}$$

with

$$f(t,x,z,\mu) \triangleq F(t,x,z,k(t,V(t,x,z,\mu)),\mu), \tag{24a}$$
$$g(t,x,z,\mu) \triangleq G(t,x,z,k(t,V(t,x,z,\mu)),\mu). \tag{24b}$$

We now state the main result of this paper.

Theorem 1. *If Assumptions 1–6 hold then, for any non-supremal rate of convergence α of the closed loop reduced order system (21), there exist $\mu^* > 0$ and a positive function $\alpha_s : (0,\mu^*) \to \mathbb{R}_+$ such that,*

1. *for $\mu < \mu^*$ the closed loop full order system (23b) is GUES about the zero state with rate $\alpha_s(\mu)$;*
2. *as μ approaches zero, $\alpha_s(\mu)$ approaches α and the corresponding gain of exponential convergence remains finite.*

We postpone the proof of this theorem to Section 8

Remark 5. The above theorem guarantees robustness of static output feedback controllers whose design is based on stabilizing the reduced order system. In other words, if the fast dynamics are sufficiently fast, the control designer can simply neglect them and yet be assured that the behavior of the closed loop full order system will be close, qualitatively and quantitatively, to the behavior of the reduced order system under the same controller.

5. Some remarks on input–output decoupling

5.1. Linear input–output decoupled systems

Here we examine the consequences of input–output decoupling for a system of the form

$$\dot{\xi} = A\xi + Bu + w_1(t),$$
$$y = C\xi + w_2(t),$$

with input u and output y, where w_1 and w_2 are vector functions of suitable dimensions. Clearly, this system is input–output decoupled iff the same is true of the following LTI system

$$\begin{aligned}\dot{\xi} &= A\xi + Bu,\\ y &= C\xi.\end{aligned} \tag{25}$$

It is readily seen that a necessary and sufficient condition for system (25) to be input–output decoupled is that the transfer function from u to y be zero, that is,

$$C(sI - A)^{-1}B \equiv 0, \tag{26}$$

where $s \in \mathbb{C}$. This condition is equivalent to the requirement that all the system Markov parameters are zero, that is

$$CA^iB = 0, \quad i = 0, 1, \ldots \tag{27}$$

The last condition is equivalent to

$$\mathscr{R}(\mathscr{C}) \subset \mathscr{N}(\mathscr{O}), \tag{28}$$

where

$$\mathscr{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \quad \mathscr{C} \triangleq [B \quad AB \quad \ldots \quad A^{n-1}B],$$

are the observability and controllable matrices, respectively, for the triple (C, A, B). In other words a necessary and sufficient condition for system (25) to be input–output decoupled is that its controllable subspace is contained in its unobservable subspace.

Finally, condition (26) is equivalent to the existence of a nonsingular matrix T such that, upon letting $\tilde{\xi} = T\xi$, one obtains

$$\begin{aligned}\dot{\tilde{\xi}} &= \widetilde{A}\tilde{\xi} + \widetilde{B}u,\\ y &= \widetilde{C}\tilde{\xi},\end{aligned} \tag{29}$$

where $\widetilde{C},\ \widetilde{A},\ \widetilde{B}$ have the following structure

$$\widetilde{C} = (\,C_1 \quad 0\,), \quad \widetilde{A} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}. \tag{30}$$

5.2. Nonlinear input–output decoupled systems

The notion of input–output decoupling has been analysed in some detail for nonlinear systems in [11, p. 124] (see also [4,12]). Here we report two of the results therein.

Consider a SISO nonlinear system of the form

$$\dot{\xi} = f(\xi) + g(\xi)u,$$
$$y = h(\xi) + w(t),$$

where $\xi(t) \in \mathbb{R}^n,\ u(t),\ y(t),\ w(t) \in \mathbb{R}$ (the multi-input multi-output case can be treated similarly). Clearly, this system is input–output decoupled iff the same is true of the following system

$$\begin{aligned}\dot{\xi} &= f(\xi) + g(\xi)u,\\ y &= h(\xi).\end{aligned} \tag{31}$$

The following holds; see [11].

Fact 1. System (31) is input–output decoupled iff

$$L_gL_f^kh(\xi) = 0 \quad \text{for } k = 0, 1, \ldots \tag{32}$$

for all $\xi \in \mathbb{R}^n$.

Note that the above condition is the nonlinear version of Markov parameter condition (27) for LTI systems.

Fact 2. If the nonlinear system (31) is input–output decoupled (and some further technical conditions are satisfied), then around each state $\xi_0 \in \mathbb{R}^n$ there is a local coordinate transformation T such that, upon letting $\tilde{\xi} = T(\xi)$, one obtains

$$\begin{aligned} \dot{\tilde{\xi}}_1 &= \tilde{f}_1(\tilde{\xi}_1), \\ \dot{\tilde{\xi}}_2 &= \tilde{f}_2(\tilde{\xi}) + \tilde{g}(\tilde{\xi})u, \\ y &= \tilde{h}(\tilde{\xi}_1), \end{aligned} \tag{33}$$

where $\tilde{\xi} = \begin{pmatrix} \tilde{\xi}_1^T & \tilde{\xi}_2^T \end{pmatrix}^T$, and $\tilde{\xi}_1,\ \tilde{\xi}_2$ are of appropriate dimensions.

This system has the same triangular structure as (29). For a nonlinear analog of condition (28) we again refer the interested reader to [11].

5.3. Stability invariance under output feedback

Here we present a lemma which will be needed in the proof of Theorem 1. Loosely speaking it states that static output feedback control does not destroy GUES if the system is input–output decoupled.

Consider the following parameterized input–output system

$$\dot{\xi} = S(t, \xi, u, \theta), \tag{34a}$$

$$y = R(t, \xi, \theta), \tag{34b}$$

where $t,\ \xi(t),\ u(t),\ y(t)$ are as before, and $\theta \in \Theta$ is the parameter. We assume that the functions $S(\cdot,\cdot,\cdot,\theta)$, $R(\cdot,\cdot,\theta)$ are continuous for each $\theta \in \Theta$. Suppose that for each $\theta \in \Theta$, the uncontrolled system

$$\dot{\xi} = S(t, \xi, 0, \theta) \tag{35}$$

has an equilibrium state $\xi_e(\theta)$ and system (34) is subject to a parameterized output feedback controller

$$u(t) = q(t, y(t), \theta),$$

where $q(\cdot,\cdot,\theta)$ is continuous. Then the corresponding closed system is described by

$$\dot{\xi} = S(t, \xi, q(t, y, \theta), \theta), \tag{36a}$$

$$y = R(t, \xi, \theta). \tag{36b}$$

The following technical condition is needed.

Condition 5.3.1. For each $\theta \in \Theta$, the function $S(\cdot,\cdot,0,\theta)$ is continuously differentiable wrt its second argument (ξ) and there exists $c \geqslant 0$ such that

$$\begin{aligned} &\|D_2 S(t, \xi, 0, \theta)\| \leqslant c, \\ &\|S(t, \xi, u, \theta) - S(t, \xi, 0, \theta)\| \leqslant c\|u\|, \\ &\|q(t, R(t, \xi, \theta), \theta)\| \leqslant c\|\xi - \xi_e(\theta)\| \end{aligned}$$

for all $t \in \mathbb{R},\ \xi \in \mathbb{R}^n,\ u \in \mathbb{R}^p$, and $\theta \in \Theta$.

We have the following result.

Lemma 2. *Suppose the parameterized system (34) is input–output decoupled for each $\theta \in \Theta$ and Condition 5.3.1 is satisfied. Then, if the uncontrolled system (35) is GUES about $\xi_e(\theta)$, the closed loop system is also GUES about $\xi_e(\theta)$ with supremal rate of convergence equal to that of the uncontrolled system.*

Proof. The appendix contains a proof. □

Remark 6. For an LTI system (25) which is input–output decoupled, one may simply show by an algebraic argument that, linear static output feedback does not change the eigenvalues of the system. To see this, recall that input–output decoupling is equivalent to the condition $C(sI - A)^{-1}B \equiv 0$ and note that

$$\begin{aligned} \det(\lambda I - A - BKC) &= \det(\lambda I - A)\det(I - (\lambda I - A)^{-1}BKC) \\ &= \det(\lambda I - A)\det(I - C(\lambda I - A)^{-1}BK) \\ &= \det(\lambda I - A). \end{aligned} \tag{37}$$

In (37) we used the matrix equality $\det(I + MN) = \det(I + NM)$, with M and N suitably dimensioned matrices (see, for example, [13]).

Remark 7. For LTI systems, input–output decoupling is necessary to guarantee that output feedback never destroys exponential stability. We make this precise as follows.

Lemma 3. *If the linear time-invariant system (25) is not input–output decoupled then, there exists a matrix $K \in \mathbb{R}^{p\times l}$ such that $A + BKC$ is not Hurwitz. Furthermore, if A is nonsingular then, K can be chosen such that $A + BKC$ is also nonsingular.*

Proof. The appendix contains a proof. □

6. LTI singularly perturbed systems

Consider the LTI singularly perturbed system (12). Its boundary layer input–output system has the form

$$\frac{dz_f}{d\tau} = A_{22}z_f + B_2u_f + A_{21}x_0, \tag{38a}$$

$$y_f = C_2z_f + C_1x_0. \tag{38b}$$

We have seen that a necessary and sufficient condition for this system to be input–output decoupled is that the transfer function from u_f to y_f be zero. Hence, an LTI singularly perturbed system satisfies the hypotheses of Theorem 1 and, hence, output feedback stabilization is robust wrt to the singular perturbation if the following hold:

$$\boxed{\begin{array}{l} A_{22} \text{ is Hurwitz} \\ C_2(sI - A_{22})^{-1}B_2 \equiv 0 \end{array}} \tag{39}$$

Remark 8. The above conditions has been used in [22] (see also [23,24]) in the framework of an example illustrating various concepts related to the *graph topology*. In that example the singularly perturbed system under consideration is linear as in (12) and it is proven that, if conditions (39) hold, then any *linear* controller which stabilizes the reduced order system is robust wrt the singular perturbation. There, robustness wrt singular perturbations is a consequence of the fact that conditions (39) ensure that as μ approaches 0, the full order system converges to the reduced order system in the graph topology. Unfortunately this powerful result cannot be used if either the system or the compensator is nonlinear.

Remark 9. Condition (39) has an interesting frequency domain interpretation. It is known [17, p. 84] that if A_{22} is nonsingular then, the transfer function matrix $G : (s, \mu) \mapsto G(s, \mu)$ for system (12) can be written as sum of two transfer function matrices in two different (complex) frequency scales, s and μs, corresponding to the time scales t and t/μ, that is

$$G(s, \mu) = G_s(s, \mu) + G_f(\mu s, \mu), \tag{40}$$

where

$$G_s(s, \mu) \triangleq C_s(\mu)(sI_n - A_s(\mu))^{-1}B_s(\mu), \tag{41a}$$

$$G_f(\mu s, \mu) \triangleq C_f(\mu)(\mu sI_m - A_f(\mu))^{-1}B_f(\mu). \tag{41b}$$

The triples $(C_s(\mu), A_s(\mu), B_s(\mu))$ and $(C_f(\mu), A_f(\mu), B_f(\mu))$ correspond to the *exact* block diagonal decomposition of system (12) in slow and fast subsystems, respectively.

Considering $\omega_f \triangleq \mu\omega$ from (40) we have

$$G(j\omega_f/\mu, \mu) = G_s(j\omega_f/\mu, \mu) + G_f(j\omega_f, \mu). \tag{42}$$

Since G_s is a strictly proper transfer matrix and $C_f,\ A_f$ and B_f are differentiable functions of μ, we obtain

$$G_s(j\omega_f/\mu, \mu) \sim O(\mu), \tag{43a}$$

$$G_f(j\omega_f, \mu) \sim G_f(j\omega_f, 0) + O(\mu). \tag{43b}$$

Since $C_f(0) = C_2,\ A_f(0) = A_{22},\ B_f(0) = B_2$, from (41)–(43) and (39) it follows that

$$G(j\omega_f/\mu, \mu) \sim O(\mu). \tag{44}$$

If $A_s(0) = A_{11} - A_{12}A_{22}^{-1}A_{21}$ and $A_f(0) = A_{22}$ are asymptotically stable, then the full order system will be asymptotically stable for sufficiently small μ. In this case we can give a frequency response interpretation of (39) by saying that the full order system (12) filters out all harmonic signals with frequency of order $1/\mu$ or larger.

We note that condition (39) is also necessary for (44). Indeed, if (44) holds, from (42), (43) one has

$$G_f(j\omega_f, 0) \sim O(\mu),$$

which implies $G_f(j\omega_f, 0) \equiv 0$.

7. Plants with fast actuators and sensors

We have already mentioned that the input–output decoupling Assumption 3 holds for singularly perturbed systems whose fast dynamics are due to the presence of sufficiently fast actuators and/or sensors. To see this, consider a plant described by the equation

$$\dot{x} = S(t, x, \tilde{u}), \tag{45}$$

where $t \in \mathbb{R}$ is the time, $x(t) \in \mathbb{R}^n$ is the state, $\tilde{u}(t) \in \mathbb{R}^q$ is the input from the actuators, and S is a continuous function. Suppose plant sensors and actuators can be modelled as follows:

$$\left.\begin{aligned} \mu\dot{z}_1 &= L_1(t, x, z_1, \mu) \\ y &= R_1(t, x, z_1, \mu) \end{aligned}\right\} \text{sensors}, \tag{46}$$

$$\left.\begin{aligned} \mu\dot{z}_2 &= L_2(t, x, z_2, u, \mu) \\ \tilde{u} &= R_2(t, x, z_2, u, \mu) \end{aligned}\right\} \text{actuators}, \tag{47}$$

where $z_1(t) \in \mathbb{R}^{m_1}$ is the sensor state; $z_2(t) \in \mathbb{R}^{m_2}$ is the actuator state; $y(t) \in \mathbb{R}^l,\ u(t) \in \mathbb{R}^p,\ \mu \in (0, \bar{\mu})$, and $L_i,\ R_i,\ i = 1, 2$, are continuous functions.

It is readily seen that the boundary layer input–output system associated with the singularly perturbed system (45)–(47) is given by

$$\frac{dz_{f1}}{d\tau} = L_1(t_0, x_0, z_{f1}, 0), \tag{48a}$$

$$\frac{dz_{f2}}{d\tau} = L_2(t_0, x_0, z_{f2}, u_f, 0), \tag{48b}$$

$$y_f = R_1(t_0, x_0, z_{f1}, 0), \tag{48c}$$

where $z_{f1}(\tau) \in \mathbb{R}^{m_1},\ z_{f2}(\tau) \in \mathbb{R}^{m_2}$, $y_f(\tau) \in \mathbb{R}^l$, and $u_f(\tau) \in \mathbb{R}^p$.

Clearly, since the "fast control" u_f affects only z_{f2} and the "fast output" depends only on z_{f1}, the input–output decoupling assumption is satisfied.

Remark 10. It is known (see [17, p. 82 and Exercise 2.8]) that for a LTI singularly perturbed description of an actuators–plant sensors system, the fast eigenvalues of the actuators are weakly (that is $O(\mu)$) observable and the fast eigenvalues of the sensors are weakly controllable. An intuitive explanation of this is that in the fast time scale the dynamics of the plant become slower and slower as μ decreases and are constant for $\mu = 0$. Thus the modes of the actuators become less and less observable "through" the plant and, dually, the modes of the sensors become less and less controllable. For $\mu = 0$ the eigenvalues of the actuators become unobservable and those of the sensors become uncontrollable; this implies that for the boundary layer input–output system the controllable subspace is contained in the unobservable subspace (condition (28)).

8. Proof of Theorem 1

We will utilize a recent result on the GUES of singularly perturbed systems [6]. To this end, we will demonstrate that the closed loop full order system (23) satisfies the following conditions.

Condition 8.1. For each $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, the equation

$$g(t, x, \bar{z}, 0) = 0 \tag{49}$$

has a unique solution

$$\bar{z} = h(t, x) \tag{50}$$

and h is continuously differentiable.

Condition 8.2. The reduced order system

$$\dot{x} = f(t, x, h(t, x), 0)$$

is GUES about the origin.

Condition 8.3. The boundary layer system

$$\frac{dz_f}{d\tau} = g(t_0, x_0, z_f, 0)$$

associated with (23b) (with (t_0, x_0) as parameter) is GUES about its equilibrium state $h(t_0, x_0)$.

Condition 8.4. The functions $f(\cdot,\cdot,\cdot,0)$, $g(\cdot,\cdot,\cdot,0)$ and h are nice.

Condition 8.5. There exist continuous functions $\kappa_f,\ \kappa_g : [0, \bar{\mu}) \to \mathbb{R}_+$, with $\kappa_f(0) = \kappa_g(0) = 0$, and a positive constant d_g such that, for all $t \in \mathbb{R},\ x \in \mathbb{R}^n,\ z \in \mathbb{R}^m$, and $\mu \in (0, \bar{\mu})$,

$$\|f(t,x,z,\mu) - f(t,x,z,0)\| \leqslant \kappa_f(\mu)(\|x\| + \|z\|),$$
$$\|g(t,x,z,\mu) - g(t,x,z,0)\| \leqslant \kappa_g(\mu)(\|x\| + \|z\|),$$
$$\kappa_g(\mu)/\mu \leqslant d_g.$$

Using Corollary 1 in [6], one may readily show that satisfaction of Conditions 8.1, 8.2–8.5 by the closed loop full order system (23b) is sufficient to guarantee that the consequences of Theorem 1 holds. We now show that the above conditions hold.

Condition 8.1 is satisfied. From definition (24b) and Assumption 1, Eq. (49) is equivalent to

$$\bar{z} = H(t, x, k(t, V(t, x, \bar{z}, 0))). \tag{51}$$

Recalling (18), it follows that if $\bar{z}$ solves (51), then we must have

$$\bar{z} = H(t, x, k(t, \bar{v}(t, x))) \triangleq h(t, x) \tag{52}$$

The fact that $\bar{z} = h(t, x)$ is a solution can be seen by noting that

$$V(t, x, h(t, x), 0) = \bar{v}(t, x) \tag{53}$$

and substituting (52) into (51). The fact that h is continuously differentiable follows from the continuous differentiability of H, k and $\bar{v}$.

Condition 8.2 is satisfied. Recalling (24a), (52) and utilizing (53) yields

$$\begin{aligned} f(t,x,h(t,x),0) &= F(t,x,h(t,x),k(t,V(t,x,h(t,x),0)),0) \\ &= F(t,x,H(t,x,k(t,\bar{v}(t,x))),k(t,\bar{v}(t,x)),0) \\ &= \bar{F}(t,x,k(t,\bar{v}(t,x))) \\ &= \bar{f}(t,x). \end{aligned}$$

Condition 8.2 is satisfied by virtue of Assumption 4.

Condition 8.3 is satisfied. Utilizing (24b), the boundary layer system associated with (23) is given by

$$\begin{aligned} \frac{dz_f}{d\tau} &= g(t_0, x_0, z_f, 0) \\ &= G(t_0, x_0, z_f, k(t_0, V(t_0, x_0, z_f, 0)), 0). \end{aligned} \tag{54}$$

This is the same as the closed loop system obtained by subjecting the boundary layer input–output system

$$\frac{dz_f}{d\tau} = G(t_0, x_0, z_f, u_f, 0),$$
$$y_f = V(t_0, x_0, z_f, 0)$$

to the output feedback $u_f = k(t_0, y_f)$. Regarding $\theta = (t_0, x_0)$ as a parameter and letting

$$S(z_f, \tilde{u}_f, \theta) \triangleq G(t_0, x_0, z_f, k(t_0, \bar{v}(t_0, x_0)) + \tilde{u}_f, 0), \tag{55}$$
$$R(z_f, \theta) \triangleq V(t_0, x_0, z_f, 0), \tag{56}$$
$$q(y_f, \theta) \triangleq k(t_0, y_f) - k(t_0, \bar{v}(t_0, x_0)) \tag{57}$$

this system is described by

$$\frac{dz_f}{d\tau} = S(z_f, \tilde{u}_f, \theta),$$
$$y_f = R(z_f, \theta)$$

subject to the output feedback $\tilde{u}_f = q(y_f, \theta)$ and we can apply Lemma 2. To demonstrate that the hypotheses of Lemma 2 hold, we first note that GUES of the uncontrolled system $(\tilde{u}_f = 0)$ around the equilibrium state $z_{fe} = h(t_0, x_0)$ follows from

Assumption 2. Assumption 3 yields the input–output decoupling requirement. Assumption 5 ensures the existence of the derivative of S wrt z_f and that

$$\begin{aligned} \|D_1S(z_f,0,\theta)\| &\leqslant M, \\ \|S(z_f,\tilde{u}_f,\theta)-S(z_f,0,\theta)\| &\leqslant M\|\tilde{u}_f\|, \\ \|q(R(z_f,\theta),\theta)\| &\leqslant M\|V(t_0,x_0,z_f,0)-V(t_0,x_0,z_{fe},0)\| \\ &\leqslant M^2\|z_f-z_{fe}\|; \end{aligned}$$

hence, Condition 5.3.1 is satisfied with $c=\max\{M,M^2\}$.

The GUES of boundary layer system (54) follows now from Lemma 2.

Condition 8.4 *is satisfied.* The satisfaction of Condition 8.4 can be straightforwardly shown using Assumption 5, Eqs. (18), (24) and (52), and the fact that a function composition of nice functions is nice (see Remark 4).

Condition 8.5 *is satisfied.* This can be shown to hold by using Assumptions 5 and 6. To illustrate, we show the first two steps:

$$\begin{aligned} \|V(t,x,z,\mu)\| &\leqslant \|V(t,x,z,\mu)-V(t,x,z,0)\|+\|V(t,x,z,0)\| \\ &\leqslant (\mu d+M)(\|x\|+\|z\|); \\ \|k(t,V(t,x,z,\mu))\| &\leqslant M\|V(t,x,z,\mu)\| \\ &\leqslant M(\mu d+M)(\|x\|+\|z\|). \end{aligned}$$

This completes the proof of Theorem 1.

Remark 11. We must point out that, even though the results summarized in Theorem 1 are stated in a "global" setting for the sake of simplicity, more general "local" results can be derived with relaxed assumptions. In this framework, for example, the GUES assumption would be replaced by a local uniform exponential stability assumption and, in the case of time-invariant systems, a local "niceness" property is implied by continuous differentiability. The interested reader is referred to [6,7] for related results.

9. Dynamical output feedback controllers

The result presented in Theorem 1 can be readily generalized to the case of dynamical output feedback control. We will consider dynamical output feedback controllers described by the following equations:

$$\dot{x}_c = L(t,x_c,y), \tag{58a}$$
$$u = k_d(t,x_c,y), \tag{58b}$$

where $t,\ y$ and u are as before, $x_c(t)\in\mathbb{R}^s$ is the controller state, and L and k_d are continuous functions.

Applying controller (58) to the reduced order system (19b), one obtains the closed loop reduced order system

$$\dot{x} = \bar{f}_d(t,x,x_c), \tag{59a}$$
$$\dot{x}_c = \bar{l}_d(t,x,x_c) \tag{59b}$$

with

$$\begin{aligned} \bar{f}_d(t,x,x_c) &\triangleq \bar{F}(t,x,k_d(t,x_c,\bar{v}(t,x))), \\ \bar{l}_d(t,x,x_c) &\triangleq L(t,x_c,\bar{v}(t,x)). \end{aligned}$$

Application of the same dynamical controller to the full order system (1) leads to the closed loop full order system:

$$\dot{x} = f_d(t,x,x_c,z,\mu), \tag{60a}$$
$$\dot{x}_c = l_d(t,x,x_c,z,\mu), \tag{60b}$$
$$\mu\dot{z} = g_d(t,x,x_c,z,\mu) \tag{60c}$$

with

$$\begin{aligned} f_d(t,x,x_c,z,\mu) &\triangleq F(t,x,z,k_d(t,x_c,V(t,x,z,\mu)),\mu), \\ l_d(t,x,x_c,z,\mu) &\triangleq L(t,x_c,V(t,x,z,\mu)), \\ g_d(t,x,x_c,z,\mu) &\triangleq G(t,x,z,k_d(t,x_c,V(t,x,z,\mu)),\mu). \end{aligned}$$

We can now substitute Assumption 4 with the following one.

Assumption 7. There exists a dynamical output feedback controller (58), with L and k_d nice functions, such that the closed loop reduced order system (58) is GUES about the origin.

Theorem 2 is the generalization of Theorem 1 we were looking for.

Theorem 2. *If Assumptions 1–3 and 5–7 hold then, for any non-supremal rate of convergence α of the closed loop reduced order system, there exist $\mu^* > 0$ and a continuous function $\alpha_s : (0, \mu^*) \to \mathbb{R}_+$ such that,*

1. *for $\mu < \mu^*$ the closed loop full order system (60) is GUES with rate $\alpha_s(\mu)$;*
2. *as μ approaches 0, $\alpha_s(\mu)$ approaches α and the gain of exponential convergence remains finite.*

Proof. To prove Theorem 2 we make use of Theorem 1. To this end, consider the following singularly perturbed system input–output system which is obtained by "opening the loop" of the closed loop system (60):

$$\dot{x} = F(t, x, z, u, \mu), \tag{61a}$$
$$\dot{x}_c = l_d(t, x, x_c, z, \mu), \tag{61b}$$
$$\mu\dot{z} = G(t, x, z, u, \mu), \tag{61c}$$
$$\hat{y} = \widehat{R}(t, x, x_c, z, \mu) \tag{61d}$$

with

$$\widehat{R}(t, x, x_c, z, \mu) \triangleq k_d(t, x_c, V(t, x, z, \mu)).$$

It can be quite readily shown that if Assumptions 1–3 and 5–7 hold, then the assumptions of Theorem 1 (namely Assumptions 1–6) are satisfied by system (61) with the static output feedback law

$$k(t, \hat{y}) = \hat{y}.$$

The details are left to the interested reader. □

Remark 12. Note that the above approach is an alternative way to show robustness of *strictly proper* dynamical output feedback controllers. More precisely, if Assumptions 1, 2 and 5–7 hold (note that the input–output decoupling assumption is not necessary in this case) and the function k_d in (58b) does not depend on y, statements 1 and 2 of Theorem 2 apply. This is due to the fact that the strict properness of the controller renders the boundary layer input–output system associated with system (61) input–output decoupled. Compare this with [14,15] where the LTI case is analysed.

10. A further remark on the input–output decoupling assumption

The next lemma states that, for LTI systems, the input–output decoupling assumption is necessary to guarantee robustness for *all* controllers which stabilize the reduced order system.

Theorem 3. *Consider a singularly perturbed LTI system described by (12) and suppose that it satisfies Assumptions 1 and 4, but does not satisfy Assumption 3. Then there exists a controller which stabilizes the corresponding reduced order system (13) but results in an unstable closed loop full order system for all μ sufficiently small.*

Proof. We will construct the above-mentioned controller sequentially, by first determining a destabilizing part for the boundary layer system and then adding a part which stabilizes the reduced order system and leaves the boundary layer system unstable. First note that if an output feedback controller

$$u = Ky + \tilde{u} \tag{62}$$

is applied to the full order system (12), the associated boundary layer input–output system is subject to the output feedback controller

$$u_f = Ky_f + \tilde{u}_f, \tag{63}$$

and the eigenvalues of the closed loop boundary layer system are the eigenvalues of $\widetilde{A}_{22} \triangleq A_{22} + B_2KC_2$. Since the boundary layer input–output system is not input–output decoupled then, in view of Lemma 3 it is always possible to choose K so that $\widetilde{A}_{22}$ is unstable. Moreover, since A_{22} is nonsingular by Assumption 1, K can be chosen so that $\widetilde{A}_{22}$ is nonsingular as well.

Application of the output feedback (62) to the full order singularly perturbed system (12) results in a new reduced order system:

$$\dot{x} = \widetilde{A}x + \widetilde{B}\tilde{u}, \tag{64a}$$
$$y = \widetilde{C}x + \widetilde{D}u, \tag{64b}$$

where the matrices $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$ depend on K.

Since the original reduced order system (13) is stabilizable via output feedback in view of Assumption 4, it is detectable and stabilizable. It is possible to show (see [5, Theorem 2]) that, if A_{22} and $\widetilde{A}_{22}$ are nonsingular, this property is not destroyed by the output feedback (62): in other words the new reduced order system (64) is also detectable and stabilizable. Hence we can construct a *strictly proper* dynamical controller for $\tilde{u}$ which stabilizes (64). The resultant overall controller generating u consists of a static part and a strictly proper dynamical part.

The LTI singularly perturbed system obtained by applying this controller to system (12) is unstable for sufficiently small μ. Indeed, since the dynamical controller is strictly proper, it is not difficult to realize that $\tilde{u}_f(\tau) \equiv \text{const}$, that is, apart from the shift in the equilibrium point, the boundary layer system is unaffected by the dynamical controller; hence it is unstable. This, in turn, implies that the full order system is unstable if μ is sufficiently small since, in this case, some of the "fast" eigenvalues of the full order system belong to the open right half complex plane [17, p. 56]. □

11. An example

Consider the one-link robotic manipulator with flexible joint illustrated in Fig. 1. The robotic link is connected to one of the ends of a massless shaft. At the other end of the shaft is a rotor to which a control torque u is applied.

If the shaft is rigid then the angular displacements θ_1 and θ_2 of the rotor and link, respectively, are always equal, that is $\theta_1 = \theta_2$, and the motion of this rigid system can be described by

$$J_T\ddot{\theta}_2 = mgl\sin\theta_2 + u, \tag{65}$$

where $J_T = J_1 + J_2$, J_1 is the moment of inertia of the rotor, J_2 is the moment of inertia of the link, m is the mass of the link and g is the gravitational acceleration constant. Suppose we can measure θ_2 and $\dot{\theta}_2$. Then a linear controller which guarantees GUES of the rigid system (65) is given by

$$u = -k_1\theta_2 - k_2\dot{\theta}_2, \tag{66}$$

provided that $k_1 > mgl$ and $k_2 > 0$.

To see this define

$$x_1 \triangleq \theta_2, \tag{67a}$$

$$x_2 \triangleq \dot{\theta}_2, \tag{67b}$$

and rewrite (65) as follows:

$$\dot{x}_1 = x_2, \tag{68a}$$

$$\dot{x}_2 = J_T^{-1}(mgl\sin x_1 + u). \tag{68b}$$

GUES of (68) with the control law (66) can be shown using the Lyapunov function $V(x) \triangleq x^TPx + 2mgl(\cos x_1 - 1)$, with

$$P = \begin{pmatrix} k_1 + \rho k_2^2/J_T & \rho k_2 \\ \rho k_2 & J_T \end{pmatrix}, \quad \rho \in (0,1).$$

Suppose now that the shaft is not rigid, but can be modelled as a parallel combination of a linear torsional spring of spring constant $\beta_s\mu^{-2} > 0$ and a linear torsional damper of damping coefficient $\beta_d\mu^{-1} > 0$. Then in general $\theta_1 \neq \theta_2$ and the mathematical description of the system is

$$J_1\ddot{\theta}_1 = T_s + u, \tag{69a}$$

$$J_2\ddot{\theta}_2 = -T_s + mgl\sin\theta_2, \tag{69b}$$

$$T_s = \mu^{-1}\beta_d(\dot{\theta}_2 - \dot{\theta}_1) + \mu^{-2}\beta_s(\theta_2 - \theta_1). \tag{69c}$$

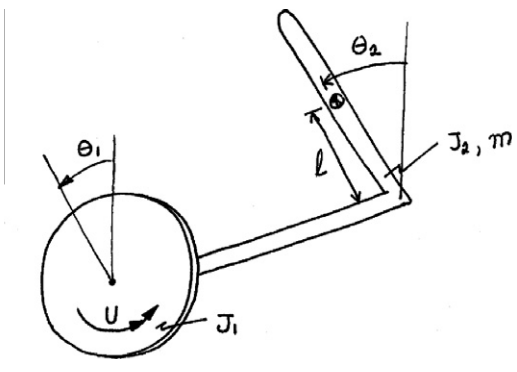

Fig. 1. One-link robotic manipulator with flexible joint.

Furthermore, suppose that the angular displacement θ_2 and velocity $\dot{\theta}_2$ of the pendulum are measured by means of sensors described as

$$\mu\dot{z}_1 = -z_1 + \theta_2, \tag{70a}$$
$$\mu\dot{z}_2 = -z_2 + \dot{\theta}_2, \tag{70b}$$
$$y_1 = z_1, \tag{70c}$$
$$y_2 = z_2. \tag{70d}$$

In order to apply Theorem 1, we must rewrite (69), (70) in the standard form (1), and then show that (i) the reduced order system is exponentially stabilized by the control law

$$u = -k_1 y_1 - k_2 y_2; \tag{71}$$

(ii) the boundary layer system is GUES and (iii) the boundary layer input–output system is input–output decoupled.

To do that, let

$$z_3 \triangleq \mu^{-2}(\theta_2 - \theta_1), \tag{72a}$$
$$z_4 \triangleq \mu^{-1}(\dot{\theta}_2 - \dot{\theta}_1). \tag{72b}$$

Eqs. (67), (72), (69) and (70) readily yield

$$\dot{x}_1 = x_2, \tag{73a}$$
$$\dot{x}_2 = J_2^{-1}(mgl \sin x_1 - \beta_s z_3 - \beta_d z_4), \tag{73b}$$
$$\mu\dot{z}_1 = -z_1 + x_1, \tag{73c}$$
$$\mu\dot{z}_2 = -z_2 + x_2, \tag{73d}$$
$$\mu\dot{z}_3 = z_4, \tag{73e}$$
$$\mu\dot{z}_4 = J_2^{-1} mgl \sin x_1 - J_P^{-1}(\beta_s z_3 + \beta_d z_4) + J_1^{-1} u, \tag{73f}$$
$$y_1 = z_1, \tag{73g}$$
$$y_2 = z_2 \tag{73h}$$

with $J_P \triangleq J_1 J_2 (J_1 + J_2)^{-1}$.

Letting $\mu = 0$, we obtain

$$\bar{z}_1 = h_1(t, x, u) = x_1, \tag{74a}$$
$$\bar{z}_2 = h_2(t, x, u) = x_2, \tag{74b}$$
$$\bar{z}_3 = h_3(t, x, u) = (\beta_s J_T)^{-1}[J_1 mgl \sin x_1 + J_2 u], \tag{74c}$$
$$\bar{z}_4 = h_4(t, x, u) \equiv 0. \tag{74d}$$

Substituting (74) into (73a), (73b), (73g), (73h) yields (68) with output

$$y_1 = x_1,$$
$$y_2 = x_2,$$

we have shown before that this system is exponentially stabilized by the output feedback control law (71).

The open loop boundary layer system associated with (73) is given by

$$\frac{dz_{f1}}{d\tau} = -z_{f1} + x_{10}, \tag{75a}$$
$$\frac{dz_{f2}}{d\tau} = -z_{f2} + x_{20}, \tag{75b}$$
$$\frac{dz_{f3}}{d\tau} = z_{f4}, \tag{75c}$$
$$\frac{dz_{f4}}{d\tau} = -J_P^{-1}(\beta_s z_{f3} + \beta_d z_{f4}) + J_2^{-1} mgl \sin x_{10} + J_1^{-1} u_0, \tag{75d}$$

which is GUES. Finally, we note that system (73) is in the actuators–plant sensors form (45)–(47) and then the input–output decoupling condition is satisfied.

Now we present some numerical results carried out with $J_1 = 1$ kg m^2, $J_2 = 1\frac{1}{3}$ kg m^2, $m = 1$ kg, $l = 1$ m, $g = 1.62$ m s^{-2}, $\beta_s = 3$ N m, $\beta_d = 3$ N m s. The control parameters are $k_1 = 2.2$, $k_2 = 3$. The initial conditions are $\theta_1(0) = \theta_2(0) = 1$ rad, $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$ rad s^{-1}, $z_1(0) = z_2(0) = 0$.

Fig. 2 presents simulation results for the reduced order system and the full order system for various values of μ; the angle of the robot link is plotted against time t. These results confirm that, for small values of the parameter μ, the full order system is exponentially stable. We see that the system is unstable for $\mu = 1.2$.

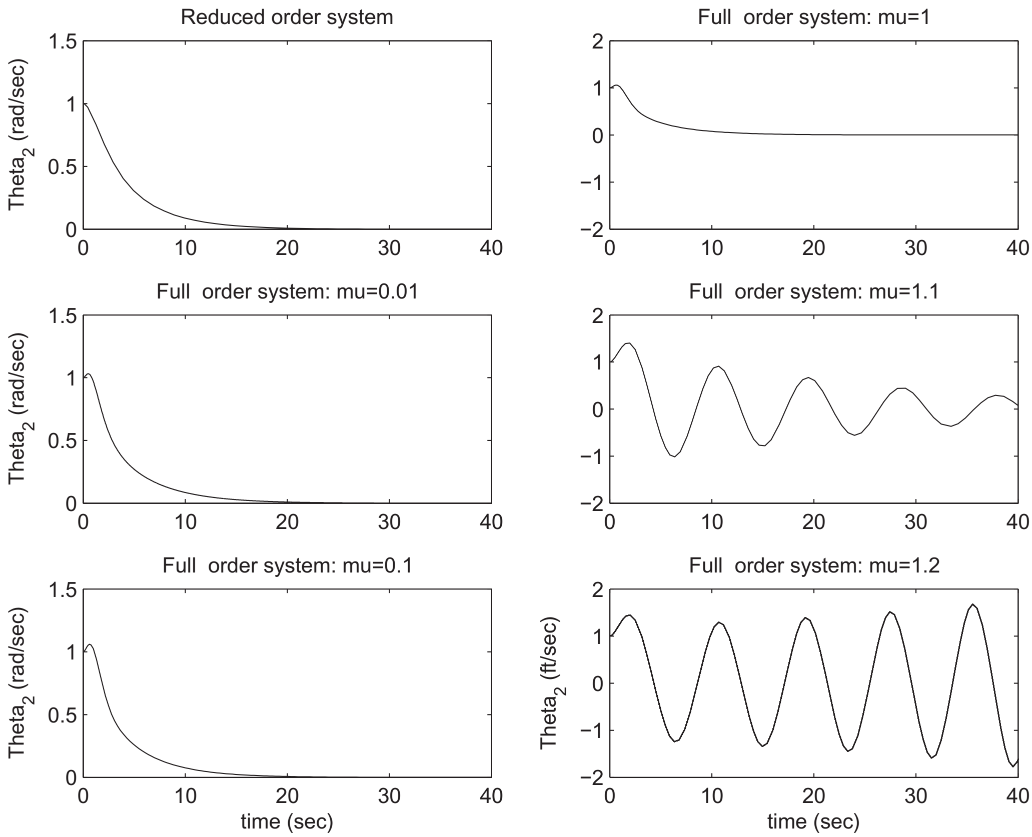

Fig. 2. Simulation results.

12. Conclusions

A novel result on the robustness of output feedback controllers for singularly perturbed systems has been presented. Besides giving a sufficient (and relatively weak) condition for robustness, our investigation presents some new insights on the mechanisms that can lead an output feedback controller tailored on the reduced order system to destabilize the full order system. Using our result we show that the practice of neglecting actuator and sensor dynamics for control design purposes has mathematical legitimacy.

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Appendix A

A.1. Proof of Lemma 2

Before proceeding with a proof of Lemma 2, we need to state another lemma. Roughly speaking, this new lemma states that a cascade of two exponentially stable systems is exponentially stable. Its proof can be derived from [7, Theorem 5 and Remark 11].

Consider a parameterized system described by

$$\dot{\xi}_1 = S_1(t, \xi_1, \theta), \tag{76a}$$

$$\dot{\xi}_2 = S_2(t, \xi_1, \xi_2, \theta) \tag{76b}$$

with $t \in \mathbb{R},\ \xi_1(t) \in \mathbb{R}^{n_1},\ \xi_2(t) \in \mathbb{R}^{n_2},\ \theta \in \Theta$ and the functions $S_1(\cdot,\cdot,\theta),\ S_2(\cdot,\cdot,\cdot,\theta)$ are continuous for each θ. This system can be regarded as a cascade of the following two subsystems

$$\dot{\xi}_1 = S_1(t, \xi_1, \theta), \tag{77a}$$

$$\dot{\xi}_2 = S_2(t, 0, \xi_2, \theta). \tag{77b}$$

We make the following regularity assumption.

Condition 13.1.1. For each θ, the derivative $D_3S_2(\cdot, 0, \cdot, \theta)$ exists and is continuous. Also, there exists a nonnegative scalar c_s such that

$$\|D_3S_2(t, 0, \xi_2, \theta)\| \leqslant c_s,$$

$$\|S_2(t, \xi_1, \xi_2, \theta) - S_2(t, 0, \xi_2, \theta)\| \leqslant c_s\|\xi_1\|$$

for all $t,\ \xi_1,\ \xi_2,\ \theta$.

We have the following result.

Lemma 4. *Suppose the two subsystems, (77a) and (77b), are GUES about the origin and regularity Condition 13.1.1 holds. Then, the cascaded system (76) is GUES about the origin with supremal rate*

$$\bar{\alpha} = \min\{\bar{\alpha}_1, \bar{\alpha}_2\},$$

where $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are the supremal rates of the two subsystems.

We can now proceed with a proof of Lemma 2.

Proof of Lemma 2. Consider, without loss of generality $\xi_e(\theta) \equiv 0$. In view of input–output decoupling, the output of the closed loop system (36) is the same as that of the following system:

$$\dot{\eta} = S(t, \eta, 0, \theta), \tag{78a}$$

$$y_a = R(t, \eta, \theta). \tag{78b}$$

More precisely, if $\xi(t_0) = \eta(t_0)$, then

$$y(t) = y_a(t), \quad \text{for all } t \geqslant t_0. \tag{79}$$

In view of this equality, the dynamics of ξ can be described by the following system

$$\dot{\eta} = S(t, \eta, 0, \theta), \tag{80a}$$

$$\dot{\xi} = S(t, \xi, k(t, \eta, \theta), \theta), \tag{80b}$$

$$k(t, \eta, \theta) = q(t, R(t, \eta, \theta), \theta) \tag{80c}$$

with the condition $\eta(t_0) = \xi(t_0)$.

System (80) is the cascade of two systems, and it is straightforward to verify that it satisfies the hypotheses of Lemma 4. Lemma 2 now follows. □

A.2. Proof of Lemma 3

Since system (25) is not input–output decoupled the matrix transfer function $H(s) \triangleq C(sI - A)^{-1}B$ has at least one element, say $h_{ij}(\cdot)$, not identically zero. Let

$$h_{ij}(s) \triangleq \frac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n},$$

where the denominator of h_{ij} is the characteristic polynomial of A.

If we apply the output feedback controller $u_j = \rho y_i$ to system (25), the set of eigenvalues of the corresponding closed loop matrix $A + BKC$ equals the set of zeros of the polynomial

$$\lambda(s) \triangleq s^n + a_1 s^{n-1} + \cdots + a_n + \rho(b_1 s^{n-1} + \cdots + b_n)$$

$$= s^n + (a_1 + \rho b_1)s^{n-1} + \cdots + (a_n + \rho b_n).$$

Since h_{ij} is not identically zero, there exists a $b_k \neq 0$; hence it is possible to choose ρ so that the coefficient $(a_k + \rho b_k)$ is negative and, hence, λ has a zero with positive real part. Thus $A + BKC$ is not Hurwitz. In addition, if A is nonsingular, the coefficient a_n is nonsingular; hence b_k can be chosen so that $a_n + \rho b_n$ is nonzero. This guarantees that $A + BKC$ is nonsingular.

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